

Bernstein type inequalities for rational functions on analytic curves and arcs

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Abstract

Borwein and Erdélyi proved a Bernstein type inequality for rational functions on the unit circle and on the real line. Here we establish asymptotically sharp extensions of their inequalities for rational functions on analytic Jordan arcs and curves. In the proofs key roles are played by Borwein-Erdélyi inequality on the unit circle, Gonchar-Grigorjan type estimate of the norm of holomorphic part of meromorphic functions and Totik's construction of fast decreasing polynomials.

Keywords: rational functions, Bernstein-type inequalities, Borwein-Erdélyi inequality, conformal mappings, Green function

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1 Introduction

Inequalities for polynomials and rational functions have rich history and numerous applications in different branches of mathematics, in particular in approximation theory (see, for example, [BE95], [MMR94] and references therein). One of the best known inequalities is Bernstein inequality for the derivative of polynomial on the interval $[-1, 1]$. This inequality was generalized and improved in several directions, for instance, instead of one interval it was considered on compact subsets of the real line (see [Tot01], [Bar92]) and on circles ([NT13], [NT14]), on lemniscates [Nag05], on systems of Jordan curves [NT05], and more recently, on analytic Jordan arcs [KN15b]. For rational functions, Bernstein type inequalities were shown for a circle [BE96], on compact subsets of the real line and circle (see [BE96], [Luk04], [DK07]). For improvements of some of such inequalities, see [Dub12] and references therein.

The aim of this paper is to extend the mentioned above Borwein-Erdélyi inequality for rational functions from a circle to an arbitrary analytic Jordan curve when poles are from a given compact set away from the curve. The obtained inequality is asymptotically sharp. As a consequence we get a similar estimate on an analytic arc as well. All the results are formulated in terms of normal derivatives of Green functions of corresponding domains. For the necessary background on potential theory, we refer to [Ran95] and [ST97]. The basic theme of this paper is similar to a recently developed method in [KN15b], but most of the details are rather different. This approach is based on using the Borwein-Erdélyi inequality as a model case, estimates for Green's functions, fast decreasing polynomials, Gonchar-Grigorjan estimate of the norm of holomorphic part of meromorphic functions, and an appropriate interpolation; for the case of an analytic arc we apply “open-up” mapping (a rational function). Although

Bernstein inequality was developed to prove inverse theorems in approximation theory, we use direct approximation theorems to establish Theorem 1.

The structure of the paper is the following: in Section 2 we formulate the statements of new results, in Section 3 auxiliary facts are collected and the main theorem is proved, in Section 4 we consider the case of one analytic arc, the last section is devoted to sharpness.

2 Statements of the new results

We denote the unit disk by $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ is called exterior of the unit disk and $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ denotes the extended complex plane. We frequently use $g_D(z, \alpha)$ for Green's function of domain D with pole at $\alpha \in D$.

Theorem 1. *Let $\Gamma \subset \mathbf{C}$ be an analytic Jordan curve, $u_0 \in \Gamma$. Let $G_1 \subset \mathbf{C}$ be the interior of Γ , G_2 be the exterior domain $G_2 := \mathbf{C}_\infty \setminus (\Gamma \cup G_1)$ and let $Z \subset G_1 \cup G_2$ be a closed set. Denote the two normals to Γ at u_0 by $n_1(u_0)$ and $n_2(u_0)$, $n_2(u_0) = -n_1(u_0)$, where $n_1(u_0)$ and $n_2(u_0)$ are pointing inward and outward respectively.*

Then, for any rational function $f(u)$ with poles in Z only, we have

$$|f'(u_0)| \leq (1 + o(1)) \|f\|_\Gamma \cdot \max \left(\sum_{\alpha} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha), \sum_{\beta} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta) \right)$$

where the sum with α (or β) is taken over all poles of f in G_1 (or in G_2 , respectively), counting multiplicities, and $o(1)$ denotes an error term that depends on Γ , u_0 and Z , tends to 0 as the total degree of f tends to infinity and is independent of f itself.

Applying an appropriate open-up mapping, we obtain

Theorem 2. *Let $\Gamma_0 \subset \mathbf{C}$ be an analytic Jordan arc, $z_0 \in \Gamma_0$ not endpoint. Denote the two normals to Γ_0 at z_0 by $n_1(z_0)$ and $n_2(z_0)$, $n_2(z_0) = -n_1(z_0)$. Let $G := \mathbf{C}_\infty \setminus \Gamma_0$ be the complementing domain and let $Z \subset G$ be a closed set. Then, for any rational function $f(z)$ with poles in Z only, we have*

$$|f'(z_0)| \leq (1 + o(1)) \|f\|_{\Gamma_0} \cdot \max \left(\sum_{\beta} \frac{\partial}{\partial n_1(z_0)} g_G(z_0, \beta), \sum_{\beta} \frac{\partial}{\partial n_2(z_0)} g_G(z_0, \beta) \right)$$

where the sum with β is taken over all poles of f in G counting multiplicities, and $o(1)$ denotes an error term that depends on Γ_0 , z_0 and Z , tends to 0 as the total degree of f tends to infinity and is independent of f itself.

Theorem 1 is asymptotically sharp as the following theorem shows.

Theorem 3. *We use the notations of Theorem 1.*

Then there exists a sequence of rational functions $\{f_n\}$ with $\deg(f_n) = n \rightarrow \infty$ with poles in Z such that

$$|f'_n(u_0)| \geq (1 - o(1)) \|f_n\|_\Gamma \cdot \max \left(\sum_\alpha \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha), \sum_\beta \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta) \right)$$

where $o(1)$ tends to 0 as $\deg(f_n) = n \rightarrow \infty$ and it depends on Γ , u_0 and Z too and the sum with α (or β) is taken over all poles of f in G_1 (or in G_2 , respectively), counting multiplicities.

A particular sequence of rational functions showing sharpness in Theorem 2 can be obtained from Theorem 2 from [KN15b] in standard way if we take a point from Z and apply a fractional-linear mapping which maps this point to infinity.

The error term $o(1)$ in Theorems 1 and 2 cannot be dropped in general, even for polynomials, see [Nag05].

3 Some background results and the proof of Theorem 1

3.1 A “rough” Bernstein type inequality

We need the following “rough” Bernstein type inequality on Jordan curves.

Proposition 4. *Let Γ be a C^2 smooth Jordan curve and $Z \subset \mathbf{C}_\infty \setminus \Gamma$ be a closed set. Then, there exists $C_1 > 0$ such that for any $u_0 \in \Gamma$ and any rational function f with poles in Z only, we have*

$$|f'(u_0)| \leq C_1 \deg(f) \|f\|_\Gamma.$$

Proof. This quickly follows from an idea attributed to Szegő (Cauchy integral formula around the point, see e.g. [Wid69], p. 133). Fix $u_0 \in \Gamma$ and and put $G_1 := \text{Int}\Gamma$, $G_2 := \mathbf{C}_\infty \setminus (\Gamma \cup \text{Int}\Gamma)$ and let $\delta_0 := \text{dist}(Z, \Gamma)$. Now fix $\alpha_j^{(0)} \in G_j$ arbitrarily, $j = 1, 2$. Let N_1 be the total order of poles of f in G_1 , N_2 be the total order of poles of f in G_2 . Obviously, $N_1 + N_2 = \deg(f)$.

Since the poles are from a compact set, it is standard (see, e.g. [KN15a], Lemma 1) that there exist $\delta_{0,1} > 0$ and $C_2 > 0$ such that if $\text{dist}(u, \Gamma) < \delta_{0,1}$, $u \in G_j$ and $\alpha_1, \alpha_2 \in Z \cap G_j$, $j = 1, 2$, then

$$\frac{1}{C_2} \leq \frac{g_{G_j}(u, \alpha_1)}{g_{G_j}(u, \alpha_2)} \leq C_2.$$

Let $\gamma := \{u : |u - u_0| = 1/\deg(f)\}$ (assume $\deg(f) > 2/\delta_0$) and we use Bernstein-Walsh estimate for f on G_1 and G_2 (see, e.g. [Ran95], Theorem 5.5.7, p. 156):

$$|f(u)| \leq \|f\|_\Gamma \exp \left(\sum_\alpha g_{G_j}(u, \alpha) \right)$$

where the sum is taken over all poles α of f in G_j , each of them appearing as many times as the order of the pole of f at α and $u \in G_j$. It is again standard (see, e.g., Lemma 1 from [KN15a]) that there exist $\delta_{0,2} > 0$ and $C_3 > 0$ such that if $\text{dist}(u, \Gamma) < \delta_{0,2}$, then $g_{G_j}(u, \alpha_j^{(0)}) \leq C_3 \text{dist}(u, \Gamma) \leq C_3 |u - u_0|$. This way we can estimate the sum in the previous displayed formula as follows

$$\begin{aligned} \sum_{\alpha} g_{G_j}(u, \alpha) &\leq \sum_{\alpha} C_2 g_{G_j}(u, \alpha_j^{(0)}) \leq \sum_{\alpha} C_2 C_3 |u - u_0| \\ &= C_2 C_3 N_j |u - u_0| = C_2 C_3 \frac{N_j}{\deg(f)} \leq C_2 C_3. \end{aligned}$$

We apply Cauchy integral formula

$$\begin{aligned} |f'(u_0)| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(u)}{(u_0 - u)^2} du \right| \leq \frac{1}{2\pi} \int_{\gamma} \left| \frac{f(u)}{(u_0 - u)^2} \right| |du| \\ &\leq \frac{1}{2\pi} 2\pi \frac{1}{\deg(f)} \frac{\|f\|_{\Gamma} \exp(C_2 C_3)}{(\deg(f))^{-2}} = \|f\|_{\Gamma} \deg(f) \exp(C_2 C_3). \end{aligned}$$

The proposition is proved with $C_1 = \exp(C_2 C_3)$ which is independent of f , $\deg(f)$ and u_0 . \square

3.2 Conformal mappings on simply connected domains

Recall, for a given curve Γ , G_1 denotes the interior of Γ and G_2 denotes the unbounded component of $\mathbf{C}_{\infty} \setminus \Gamma$.

As earlier, $\mathbb{D} = \{v : |v| < 1\}$ and $\mathbb{D}^* = \{v : |v| > 1\} \cup \{\infty\}$. With these notations, $\partial G_1 = \partial G_2$. Using Kellogg-Warschawski theorem (see e.g. [Pom92] p. 49, Theorem 3.6), if the boundary is $C^{1,\alpha}$ smooth, then the Riemann mappings of \mathbb{D}, \mathbb{D}^* onto G_1, G_2 respectively and their derivatives can be extended continuously to the boundary.

Under analyticity assumption, we can compare the Riemann mappings as follows.

Proposition 5. *Let $u_0 \in \partial G_1 = \partial G_2$ be fixed. Then there exist two Riemann mappings $\Phi_1 : \mathbb{D} \rightarrow G_1$, $\Phi_2 : \mathbb{D}^* \rightarrow G_2$ such that $\Phi_j(1) = u_0$ and $|\Phi_j'(1)| = 1$, $j = 1, 2$.*

If $\partial G_1 = \partial G_2$ is analytic, then there exist $0 \leq r_1 < 1 < r_2 \leq \infty$ such that Φ_1 extends to $D_1 := \{v : |v| < r_2\}$, $G_1^+ := \Phi_1(D_1)$ and $\Phi_1 : D_1 \rightarrow G_1^+$ is a conformal bijection, and similarly, Φ_2 extends to $D_2 := \{v : |v| > r_1\} \cup \{\infty\}$, $G_2^+ := \Phi_2(D_2)$ and $\Phi_2 : D_2 \rightarrow G_2^+$ is a conformal bijection.

This proposition is Proposition 7 in [KN15b].

From now on, we fix such two conformal mappings. These mappings and domains are depicted on figure 1. We may assume that D_1 and D_2 are of positive distance from one another (by slightly decreasing r_1 , if necessary).

Denote the normal vector (of unit length) to Γ at u_0 or $\partial \mathbb{D}$ at 1 pointing inward by $n_1(u_0)$ or $n_1(1)$ respectively. Similarly, the outward normal vectors are denoted by $n_2(u_0)$ and $n_2(1)$.

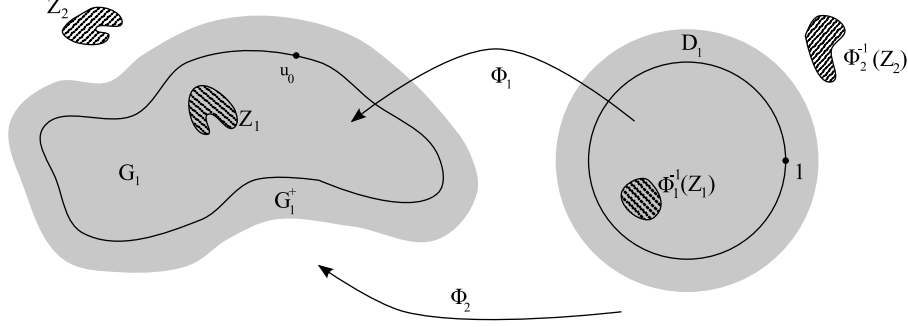


Figure 1: The two conformal mappings Φ_1 , Φ_2 , the domain D_1 and the possible location of poles

Proposition 6. *The following hold for arbitrary $\alpha \in G_1$, $\beta \in G_2$ with $\alpha' := \Phi_1^{-1}(\alpha)$, $\beta' := \Phi_2^{-1}(\beta)$ if $\beta' \neq \infty$*

$$\begin{aligned} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha) &= \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha') = \frac{1 - |\alpha'|^2}{|1 - \alpha'|^2}, \\ \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta) &= \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta') = \frac{|\beta'|^2 - 1}{|1 - \beta'|^2}, \end{aligned}$$

and if $\beta' = \infty$, then

$$\frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta) = \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \infty) = 1.$$

This proposition is a slight generalization of Proposition 8 from [KN15b] with the same proof.

3.3 Fast decreasing rational functions with prescribed poles

The next result is based on a general construction of fast decreasing polynomials by Totik, see [Tot10], Corollary 4.2 and Theorem 4.1 too.

Theorem 7. *Let $\tilde{K} \subset \mathbb{C}$ be a compact set, $\tilde{u} \in \partial\tilde{K}$ be a boundary point. Assume that \tilde{K} satisfies the touching-outer-disk condition, that is, there exists a closed disk (with positive radius) such that its intersection with \tilde{K} is $\{\tilde{u}\}$. Let $Z_2^* \subset \mathbb{C}_\infty \setminus \tilde{K}$ be a finite set and $\tau > 1$.*

Then there exist $C_4, C_5 > 0$ with the following properties. For any given multiplicity function $m : Z_2^ \rightarrow \{1, 2, \dots\}$ introduce $\tilde{n} := \sum_{\alpha} m(\alpha)$ where the sum is taken for $\alpha \in Z_2^*$ and there exists a rational function Q such that $Q(\tilde{u}) = 1$, $\|Q\|_{\tilde{K}} \leq 1$, Q has poles at the points of Z_2^* only and the order of the pole of Q at $\alpha \in Z_2^*$ is at most $m(\alpha)$ and for $u \in \tilde{K}$ we have*

$$|Q(u)| \leq C_4 \exp(-C_5 \tilde{n} |u - \tilde{u}|^\tau).$$

Proof. Roughly speaking, we transform for each $\alpha \in Z_2^*$ the fast decreasing polynomials to move their poles from ∞ to α and multiply them together.

Let $\alpha \in Z_2^*$ be fixed and let $\tilde{K}_\alpha := \left\{ \frac{1}{u-\alpha} : u \in \tilde{K} \right\}$. Obviously, \tilde{K}_α is a compact set and also satisfies the touching-outer-disk condition at $\frac{1}{\tilde{u}-\alpha}$. Using theorem 4.1 on page 2065 from [Tot10], we know that there are $d_{\alpha,\tau}, D_{\alpha,\tau} > 0$ depending on α, \tilde{K} and τ only and we get a polynomial Q_α such that $Q_\alpha\left(\frac{1}{\tilde{u}-\alpha}\right) = 1$, $\|Q_\alpha\|_{\tilde{K}_\alpha} = 1$, $\deg(Q_\alpha) \leq m(\alpha)$ and for $u \in \tilde{K}$

$$\begin{aligned} \left| Q_\alpha\left(\frac{1}{u-\alpha}\right) \right| &\leq D_{\alpha,\tau} \exp\left(-d_{\alpha,\tau} m(\alpha) \left| \frac{1}{u-\alpha} - \frac{1}{\tilde{u}-\alpha} \right|^\tau\right) \\ &\leq D_{\alpha,\tau} \exp\left(-d_{\alpha,\tau} m(\alpha) \operatorname{dist}\left(\partial\tilde{K}, Z_2^*\right)^{-2\tau} |u - \tilde{u}|^\tau\right). \end{aligned}$$

Let $Q(u) := \prod_{\alpha \in Z_2^*} Q_\alpha\left(\frac{1}{u-\alpha}\right)$. We immediately have $Q(\tilde{u}) = 1$, $\|Q\|_{\tilde{K}} = 1$, and Q has poles at the points of Z_2^* only, the order of the pole of Q at $\alpha \in Z_2^*$ is at most $m(\alpha)$. We multiply the estimates above together, so

$$\begin{aligned} |Q(u)| &\leq \left(\prod_{\alpha} D_{\alpha,\tau} \right) \exp\left(-\operatorname{dist}\left(\partial\tilde{K}, Z_2^*\right)^{-2\tau} |u - \tilde{u}|^\tau \sum_{\alpha} d_{\alpha,\tau} m(\alpha)\right) \\ &\leq C_4 \exp\left(-\left(\min_{\alpha} d_{\alpha,\tau}\right) \operatorname{dist}\left(\partial\tilde{K}, Z_2^*\right)^{-2\tau} n |u - \tilde{u}|^\tau\right) \\ &= C_4 \exp(-C_5 n |u - \tilde{u}|^\tau) \end{aligned}$$

where $C_4 = \prod_{\alpha} D_{\alpha,\tau}$ and $C_5 = (\min_{\alpha} d_{\alpha,\tau}) \operatorname{dist}\left(\partial\tilde{K}, Z_2^*\right)^{-2\tau}$. \square

3.4 Outer touching circles and other quantities

In this subsection we construct some auxiliary sets and the fast decreasing rational functions. We also use $\arg(z)$ in the following sense: if $z \neq 0$, then $\arg(z) = z/|z|$ and $\arg(0) = 0$.

Let $\Gamma_1 = \{v : |v| = 1 + \delta_1\}$ where $0 < \delta_1 < r_2 - 1$, $\delta_1 < 1/2$ is fixed. It is important that δ_1 depends on G_1 only.

Let $D_3 := \{v : |v - 2| < 1\}$, this disk touches the unit disk at 1.

For every $\delta_{2,3} \in (0, \delta_1]$, $\{v : |v| = 1 + \delta_{2,3}\} \cap \partial D_3$ consists of exactly two points, $v_1^* = v_1^*(\delta_{2,3})$ and $v_2^* = v_2^*(\delta_{2,3})$. It is easy to see that the length of the two arcs of $\{v : |v| = 1 + \delta_{2,3}\}$ lying between v_1^* and v_2^* are different, therefore, by reindexing them, we can assume that the shorter arc is going from v_1^* to v_2^* counterclockwise. Elementary geometric considerations show that for all v , $1 \leq |v| \leq 1 + \delta_{2,3}$ with $\arg v \in \{\arg v_j^*(\delta_{2,3}) : j = 1, 2\}$, we have (since $\delta_{2,3} < 1$)

$$\frac{1}{2} \sqrt{\delta_{2,3}} \leq |v - 1| \leq 2 \sqrt{\delta_{2,3}}. \quad (1)$$

Let

$$K_v^* := \{v : |v| \leq 1 + \delta_1\} \setminus D_3.$$

Obviously, this K_v^* is a compact set and satisfies the touching-outer-disk condition at 1 of Theorem 7.

Consider

$$K_u^* := \Phi_2[K_v^* \cap \mathbb{D}^*] \cup \Phi_1[K_v^* \cap \mathbb{D}^*] \cup G_1.$$

This is a compact set and also satisfies the touching-outer-disk condition at $u_0 = \Phi_2(1)$ of Theorem 7. Obviously, $\partial G_2 \subset K_u^*$, $G_1 \subset K_u^*$, $u_0 \in K_u^*$ and if $v \in K_v^*$, then $\Phi_1(v) \in K_u^*$ and $\Phi_2(v) \in K_u^*$ too.

Take any finite set $Z_2^* \subset Z_2$ with arbitrary multiplicity function such that the total multiplicity is at most $N_3 := n^{3/4}$. Now applying Theorem 7, there exists a fast decreasing rational function for K_u^* at u_0 and its degree is N_4 , where $N_4 \leq N_3$. Denote the poles of Q by $\zeta_1, \dots, \zeta_{N_4} \in Z_2^*$ counting multiplicities.

Then $\deg(Q) = N_4 \leq N_3$, $Q(u_0) = 1$, $|Q(u)| \leq 1$ on if $u \in K_u^*$, moreover

$$|Q(u)| \leq C_4 \exp(-C_5 N_3 |u - u_0|^\tau). \quad (2)$$

Note that C_4 and C_5 depend on Z_2^* and K_u^* but they are independent of the multiplicity function and N_3 and n .

For simplicity, we put $\tau := 4/3$.

3.5 The proof of Theorem 1

3.5.1 Decomposition of the rational function

It is easy to decompose f into sum of rational functions, that is,

$$f = f_1 + f_2$$

where f_1 is a rational function with poles in Z_1 , $f_1(\infty) = 0$ and f_2 is a rational function with poles in Z_2 . This decomposition is unique. Put $N_1 := \deg(f_1)$, $N_2 := \deg(f_2)$, then $N_1 + N_2 = n$. Denote the poles of f_1 by α_j , $j = 1, \dots, N_1$ and the poles of f_2 by β_j , $j = 1, \dots, N_2$ (with counting the orders of the poles).

Now fix

$$\delta_{2,1} := \frac{1}{2n}, \quad \delta_{2,3} := \min(n^{-2/3}, \delta_1).$$

We use a Gonchar-Grigorjan type estimate for f_2 on G_1 (see Theorem 1 in [KN15a]) so there exists $C_6 = C_6(G_1) > 0$ such that we have

$$\|f_2\|_\Gamma \leq C_6(G_1) \log(n) \|f\|_\Gamma. \quad (3)$$

Obviously, we have

$$\|f_1\|_\Gamma \leq (1 + C_6(G_1) \log(n)) \|f\|_\Gamma. \quad (4)$$

Consider

$$\varphi_1(v) := f_1(\Phi_1(v)).$$

We know that

$$\|\varphi_1\|_{\partial \mathbb{D}} = \|f_1\|_{\partial G_2} \quad (5)$$

and $|\varphi_1'(1)| = |f_1'(u_0)|$.

We use the fast decreasing rational function Q from Subsection 3.4.

We decompose “the essential part of” φ_1 as follows

$$Q \circ \Phi_1 \cdot \varphi_1 = \varphi_{1r} + \varphi_{1e} \quad (6)$$

where φ_{1r} is a rational function, $\varphi_{1r}(\infty) = 0$ and φ_{1e} is holomorphic in \mathbb{D} . We use the decomposition

$$(Qf) \circ \Phi_1 = (Q(f_1 + f_2)) \circ \Phi_1 = \varphi_{1r} + \varphi_{1e} + (Qf_2) \circ \Phi_1.$$

We apply the Gonchar-Grigorjan type estimate (see Theorem 1 [KN15a]) again for φ_{1e} on \mathbb{D} , this way the following sup norm estimate holds

$$\|\varphi_{1e}\|_{\partial\mathbb{D}} \leq C_6(\mathbb{D}) \log(n) \|Q \circ \Phi_1 \cdot \varphi_1\|_{\partial\mathbb{D}} \leq C_6(\mathbb{D}) \log(n) \|\varphi_1\|_{\partial\mathbb{D}} \quad (7)$$

where $C_1(\mathbb{D})$ is a constant independent of φ_1 .

Furthermore, we can estimate $\varphi_{1e}(v)$ on $v \in D_1 \setminus \mathbb{D}$ as follows

$$|\varphi_{1e}(v)| = |(Q \cdot f_1) \circ \Phi_1(v) - \varphi_{1r}(v)| \leq |(Q \cdot f_1) \circ \Phi_1(v)| + |\varphi_{1r}(v)|. \quad (8)$$

We also need to estimate Q outside G_1 as follows.

Using Bernstein-Walsh estimate, we can write for $v \in D_1 \setminus \mathbb{D}$

$$|Q(\Phi_1(v))| \leq 1 \cdot \exp \left(\sum_{j=1}^{N_4} g_{G_2}(\Phi_1(v), \zeta_j) \right)$$

where we use that the set $\Phi_1(D_1 \setminus \mathbb{D})$ is bounded,

$$C_7 := \sup \{g_{G_2}(\Phi_1(v), \beta) : v \in D_1 \setminus \mathbb{D}, \beta \in Z_2\} < \infty.$$

Therefore, for all $v \in D_1 \setminus \mathbb{D}$,

$$|(Q \cdot f_1) \circ \Phi_1(v)| \leq e^{C_7 N_4} \|f_1\|_{\Gamma}.$$

This way we can continue (8) and we use $u = \Phi_1(v)$ and that φ_{1r} is a rational function with no poles outside \mathbb{D} and the maximum principle for φ_{1r}

$$\leq e^{C_7 N_4} |f_1(u)| + \|\varphi_{1r}\|_{\partial\mathbb{D}} \leq e^{C_7 N_4} \|f_1\|_{\Gamma} + \|\varphi_1\|_{\partial\mathbb{D}} + \|\varphi_{1e}\|_{\partial\mathbb{D}}$$

and here we used that f_1 has no pole in G_2 and the maximum principle. We can estimate these three sup norms with the help of (4) and (5), (4) and (7), (5), (4). Hence we have for $v \in D_1 \setminus \mathbb{D}$

$$\begin{aligned} |\varphi_{1e}(v)| &\leq (e^{C_7 N_4} + 1 + C_6(\mathbb{D}) \log(n)) (1 + C_6(G_1) \log(n)) \|f\|_{\Gamma} \\ &= O(\log(n) e^{C_7 N_4}) \|f\|_{\Gamma}. \end{aligned} \quad (9)$$

3.5.2 Approximating the interior error function

In this subsection we construct an approximation to φ_{1e} which is holomorphic on a larger set containing \mathbb{D} . Later we will use properties (17) and (18) only.

The approximation is done by interpolating φ_{1e} as follows. φ_{1e} is holomorphic in $D_1 = \{v \in \mathbf{C} : |v| < 1 + \delta_1\}$.

Put

$$N_5 := N_2 + N_4 + \lceil n^{4/5} \rceil,$$

where N_2 is $\deg(f_2)$ and n is $\deg(f)$.

For simplicity, put

$$\alpha'_j := \Phi_1^{-1}(\alpha_j) \text{ and } \beta'_k := \Phi_2^{-1}(\beta_k)$$

where $j = 1, \dots, N_1$ and $k = 1, \dots, N_2$. Introduce q for the interpolation as follows

$$q(v) := \prod_{j=1}^{N_2} \frac{1 - \overline{\beta'_j} v}{v - \beta'_j} \cdot \prod_{j=1}^{N_5 - N_2 - 2} \frac{1 - \overline{\gamma_j} v}{v - \gamma_j} \cdot (v - 1)^2 \quad (10)$$

where γ_j , $j = 1, \dots, N_5 - N_2 - 2$ are from $\Phi_2^{-1}(Z_2)$, the first N_4 coincide with the ζ_j 's (i.e. $\zeta_j = \gamma_j' = \Phi_2(\gamma_j)$ for $j = 1, \dots, N_4$) and are arbitrary anyway.

Modulus of q can be written as follows (when $|\xi| \geq 1$)

$$|q(\xi)| = \exp \left(2 \log |\xi - 1| + \sum_{j=1}^{N_2} \log |B(\beta_j', \xi)| + \sum_{k=1}^{N_5 - N_2 - 2} \log |B(\gamma_k, \xi)| \right) \quad (11)$$

where $B(a, v) = \frac{1 - \bar{a}v}{v - a}$.

Note that if $|\xi| = 1$, then $|q(\xi)| \leq 4$.

On $D_1 \setminus \mathbb{D}$, $|q(\cdot)|$ can be estimated from below as follows. We know that D_1 and $\Phi_2^{-1}(Z_2)$ are disjoint and they are of positive distance from one another. Therefore there exists $C_8 > 0$ such that for all $\beta \in \Phi_2^{-1}(Z_2)$ and $v \in D_1 \setminus \mathbb{D}$, the modulus of the derivative of the Blaschke factor at v with pole at β is at least C_8 : $|B'(\beta, v)| \geq C_8$. This implies that for all $\beta \in \Phi_2^{-1}(Z_2)$ and $v \in D_1 \setminus \mathbb{D}$

$$\exp g_{\mathbb{D}^*}(v, \beta) \geq 1 + C_8 (|v| - 1). \quad (12)$$

Moreover there is a similar upper estimate. That is, there exists $C_9 > 0$ such that for all $\beta \in \Phi_2^{-1}(Z_2)$ and $v \in D_1 \setminus \mathbb{D}$

$$\exp g_{\mathbb{D}^*}(v, \beta) \leq 1 + C_9 (|v| - 1). \quad (13)$$

Multiplying (12) for all Blaschke factors (appearing in (10)), we obtain for $|v| = 1 + \delta_1$,

$$|q(v)| \geq (1 + C_8 \delta_1)^{N_5 - 2} \delta_1^2. \quad (14)$$

We also have a sharper lower estimate:

$$|q(v)| \geq (1 + C_8 \delta_1)^{N_5 - N_2 - 2} \delta_1^2 \prod_{j=1}^{N_2} |B(\beta_j', v)|. \quad (15)$$

We define the approximating rational function (see e.g. [Wal65] chapter VIII)

$$r_{1, N_5}(v) := \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\varphi_{1e}(\xi)}{q(\xi)} \frac{q(v) - q(\xi)}{v - \xi} d\xi.$$

It is well known that r_{1, N_5} does not depend on Γ_1 . Since 1 is a double zero of q , therefore r_{1, N_5} and r'_{1, N_5} coincide there with φ_{1e} and φ'_{1e} respectively.

The error of the approximating rational function r_{1, N_5} can be written as

$$\varphi_{1e}(v) - r_{1, N_5}(v) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\varphi_{1e}(\xi)}{\xi - v} \frac{q(v)}{q(\xi)} d\xi, \quad (16)$$

where $\Gamma_1 = \{v \in \mathbf{C} : |v| = 1 + \delta_1\}$ and recall that φ_{1e} is holomorphic in $\{|v| < 1 + \delta_1\}$ and is continuous on $\{|v| \leq 1 + \delta_1\}$.

For $v \in \mathbb{D}$ the error can be estimated as follows

$$\begin{aligned} |\varphi_{1e}(v) - r_{1, N_5}(v)| &= \frac{1}{2\pi} \int_{\Gamma_1} \frac{1}{|\xi - v|} |d\xi| \cdot \|\varphi_{1e}\|_{D_1} 4 \frac{1}{(1 + C_8 \delta_1)^{N_5 - 2}} \frac{1}{\delta_1^2} \\ &\leq 4 \frac{1 + \delta_1}{\delta_1^3} \|\varphi_{1e}\|_{D_1} \frac{1}{(1 + C_8 \delta_1)^{N_5 - 2}} = 4 \frac{1 + \delta_1}{\delta_1^3} \frac{O(\log(n) e^{C_7 N_4})}{(1 + C_8 \delta_1)^{N_5 - 2}} \|f\|_{\Gamma} \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$, because $N_5/N_4 \rightarrow \infty$ and δ_1 is fixed, that is

$$\frac{e^{C_7 N_4}}{(1 + C_8 \delta_1)^{N_5 - 2}} = \exp(C_7 N_4 - \log(1 + C_8 \delta_1) N_5 (1 + o(1))) \rightarrow 0.$$

So, r_{1, N_5} is a rational function with poles in $\Phi_2^{-1}(Z_2)$ only and we know that

$$\|\varphi_{1e} - r_{1, N_5}\|_{\partial \mathbb{D}} = o(1) \|f\|_{\Gamma} \quad (17)$$

where $o(1)$ is independent of f and depends only on Γ and tends to 0 as $n \rightarrow \infty$, furthermore

$$\varphi'_{1e}(1) = r'_{1, N_5}(1). \quad (18)$$

3.5.3 Approximating the term with poles outside

Now we interpolate and approximate $(Q \cdot f_2) \circ \Phi_1$.

We have the following description of the growth of Green's function.

Lemma 8. *There exists $C_{10} > 0$ depending on G_2 only and is independent of n and f such that for all $1 \leq |v| \leq 1 + \delta_1$ and $\beta \in Z_2$ we have*

$$g_{G_2}(\Phi_1(v), \beta) \leq C_{10}(|v| - 1). \quad (19)$$

Proof. We can express Green's function in the following way for $u \in G_2$,

$$g_{G_2}(u, \beta) = \log |B(\beta, \Phi_2^{-1}(u))|$$

where $B(\beta, v) = \frac{1 - \bar{\beta}v}{v - \beta}$ is Blaschke factor. Hence

$$g_{G_2}(\Phi_1(v), \beta) = \log |B(\beta, \Phi_2^{-1} \circ \Phi_1(v))|.$$

Here we use that there exists C_{11} such that for all $1 \leq |v| \leq 1 + \delta_1$,

$$\left| (\Phi_2^{-1} \circ \Phi_1(\cdot))'(v) \right| \leq C_{11}$$

and there exists C_{12} such that for all $1 \leq |v| \leq 1 + \delta_1$, and $\beta \in Z_2$,

$$|B(\beta, v)'| \leq C_{12}.$$

Finally, the directional derivative of $g_{\mathbb{D}^*}$ at v from direction $v_1 = v/|v|$ can be estimated as

$$\frac{\partial}{\partial v_1} g_{\mathbb{D}^*}(v, \beta) \leq \left| B(\beta, \cdot)'|_{\Phi_2^{-1} \circ \Phi_1(v)} \right| \left| (\Phi_2^{-1} \circ \Phi_1(\cdot))'(v) \right| \leq C_{11} C_{12}$$

and integrating it along the radial ray $[v_1, v]$, we obtain (19). □

Now we give the approximating rational function as follows

$$r_{2, N_5}(v) := \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(Q \cdot f_2) \circ \Phi_1(\xi) q(v) - q(\xi)}{q(\xi)} \frac{1}{v - \xi} d\xi$$

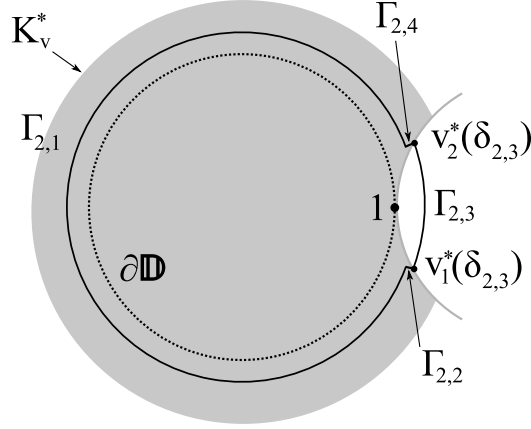


Figure 2: The integration curve Γ_2

where Γ_2 can be arbitrary as long as $\mathbb{D} \subset \text{Int}\Gamma_2$ and $\Gamma_2 \subset D_1$, and q is defined above. We remark that we use the same interpolating points, but we need a different Γ_2 for the error estimate.

Now we construct Γ_2 for the estimate and investigate the error. We use $\delta_{2,1} = 1/(2n)$, $\delta_{2,3} = \min(n^{-2/3}, \delta_1)$. We give four Jordan arcs that will make up Γ_2 . Let $\Gamma_{2,3}$ be the (shorter, circular) arc between $v_1^*(\delta_{2,3})$ and $v_2^*(\delta_{2,3})$, $\Gamma_{2,1}$ be the longer circular arc between $v_1^*(\delta_{2,3})^{\frac{1+\delta_{2,1}}{1+\delta_{2,3}}}$ and $v_2^*(\delta_{2,3})^{\frac{1+\delta_{2,1}}{1+\delta_{2,3}}}$,

$$\Gamma_{2,2} : = \{v : 1 + \delta_{2,1} \leq |v| \leq 1 + \delta_{2,3}, \arg v = \arg(v_1^*(\delta_{2,3}))\}$$

and similarly

$$\Gamma_{2,4} : = \{v : 1 + \delta_{2,1} \leq |v| \leq 1 + \delta_{2,3}, \arg v = \arg(v_2^*(\delta_{2,3}))\}$$

be the two segments connecting $\Gamma_{2,1}$ and $\Gamma_{2,3}$. Finally let Γ_2 be the union of $\Gamma_{2,1}$, $\Gamma_{2,2}$, $\Gamma_{2,3}$ and $\Gamma_{2,4}$. The figure 2 depicts these arcs and K_v^* defined above.

We estimate the error of r_{2,N_5} to $(Q \cdot f_2) \circ \Phi_1$ on each integral separately:

$$\begin{aligned} (Q \cdot f_2) \circ \Phi_1(v) - r_{2,N_5}(v) &= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(Q \cdot f_2) \circ \Phi_1(\xi)}{\xi - v} \frac{q(v)}{q(\xi)} d\omega \\ &= \frac{1}{2\pi i} \left(\int_{\Gamma_{2,1}} + \int_{\Gamma_{2,2}} + \int_{\Gamma_{2,3}} + \int_{\Gamma_{2,4}} \right). \end{aligned}$$

For the first term, we use Bernstein-Walsh estimate for the rational function f_2 on G_2 and the fast decreasing rational function Q as follows. If $v \in \Gamma_{2,1}$, then with (19), $g_{G_2}(\Phi_1(v), \beta) \leq C_{10}\delta_{2,1} = C_{10}/(2n)$ (uniformly in $\beta \in Z_2$), therefore

$$\begin{aligned} |f_2(\Phi_1(v))| &\leq \|f_2\|_{\Gamma} \exp\left(n \frac{C_{10}}{2n}\right) \leq \|f\|_{\Gamma} C_6(G_1) \log(n) e^{C_{10}/2} \\ &= O(\log(n)) \|f\|_{\Gamma} \end{aligned}$$

where we used (3). Now we use the fast decreasing property of Q as follows.

We use

$$C_{13} := \inf \{ |\Phi'_1(v)| : v \in D_1 \} > 0.$$

Hence, if $v \in \Gamma_{2,1} \cup \Gamma_{2,2} \cup \Gamma_{2,4}$, then by (1), $|v - 1| \geq 1/2 \sqrt{\delta_{2,3}}$, and then

$$|u - u_0| \geq \frac{C_{13}}{2} \sqrt{\delta_{2,3}} \quad (20)$$

where $u = \Phi_1(v)$ and $u_0 = \Phi_1(1)$ and therefore the fast decreasing rational function Q is small, see (2), in other words,

$$|Q(\Phi_1(v))| \leq C_4 \exp \left(-C_5 N_3 \left| C_{13}/2 \sqrt{\delta_{2,3}} \right|^\tau \right). \quad (21)$$

Therefore we can write

$$\begin{aligned} |(Q \cdot f_2)(\Phi_1(v))| &\leq O(\log(n)) \|f\|_{\partial G_2} C_4 \exp \left(-C_5 N_3 \left| C_{13}/2 \sqrt{\delta_{2,3}} \right|^\tau \right) \\ &\leq O \left(\frac{\log(n)}{\exp(C_5 (C_{13}/2)^\tau N_3 n^{-\tau/3})} \right) \|f\|_\Gamma =: E_{2,1} \|f\|_\Gamma \end{aligned}$$

where the coefficient tends to 0 because, with the definition of $\delta_{2,3}$, we can write $N_3 \delta_{2,3}^{\tau/2} = n^{3/4} \min(n^{-\tau/3}, \delta_1^{\tau/2})$ and $n^{3/4} n^{-\tau/3} \rightarrow \infty$ since $\tau = 4/3$.

Integrating along $\Gamma_{2,1}$, we can write for $v \in \mathbb{D}$

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{\Gamma_{2,1}} \frac{(Q \cdot f_2) \circ \Phi_1(\xi)}{\xi - v} \frac{q(v)}{q(\xi)} d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{\Gamma_{2,1}} \frac{1}{|\xi - v|} E_{2,1} \|f\|_\Gamma 4 \frac{1}{(1 + C_8 \delta_{2,1})^{N_2} \delta_{2,1}^2} |d\omega| \\ &\leq \frac{2}{\pi} \frac{2\pi(1 + \delta_{2,1})}{(1 + C_8 \delta_{2,1})^{N_2} \delta_{2,1}^3} E_{2,1} \|f\|_\Gamma = O(n^3) E_{2,1} \|f\|_\Gamma \end{aligned}$$

here we used $\delta_{2,1} = 1/(2n)$ and $n^3 E_{2,1} \rightarrow 0$.

We estimate the third term, the integral on $\Gamma_{2,3}$, as follows for $v \in \mathbb{D}$

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{\Gamma_{2,3}} \frac{(Q \cdot f_2) \circ \Phi_1(\xi)}{\xi - v} \frac{q(v)}{q(\xi)} d\omega \right| \\ &\leq \frac{1}{2\pi} \int_{\Gamma_{2,3}} 4 \frac{1}{|\xi - v|} |(Q \cdot f_2)(\Phi_1(\xi))| \frac{1}{|q(\xi)|} |d\xi|. \quad (22) \end{aligned}$$

Here, $|\xi| = 1 + \delta_{2,3}$ and $|v - \xi| \geq \delta_{2,3}$. Roughly speaking, f_2 grows and this time Q grows too and only $|q(\xi)|^{-1}$ decreases. We are going to estimate the growths with the help of Bernstein-Walsh estimate and Blaschke factors.

For simplicity, put $F(\xi) := \Phi_2^{-1} \circ \Phi_1(\xi)$, hence $F(1) = 1$, $F'(1) = 1$. This latter holds because $\Phi_1(1) = \Phi_2(1) = u_0$. Moreover, $\Phi_1(\partial\mathbb{D}) = \Phi_2(\partial\mathbb{D}) = \Gamma$, hence $\arg \Phi'_1(1) = \arg \Phi'_2(1)$ and together with $|\Phi'_1(1)| = |\Phi'_2(1)| = 1$, it implies that $\Phi'_1(1) = \Phi'_2(1)$. Therefore $(\Phi_2^{-1} \circ \Phi_1)'(1) = 1$. Moreover, $\exp g_{G_2}(\Phi_1(\xi), \Phi_2(\gamma)) = B(\gamma, F(\xi))$ (if $|\gamma| > 1$).

First, Bernstein-Walsh estimate for f_2 on G_2 yields for $\xi \in D_1 \setminus \mathbb{D}$ (and in particular, if $\xi \in \Gamma_{2,3}$) that

$$\begin{aligned} |f_2(\Phi_1(\xi))| &\leq \|f_2\|_{\Gamma} \exp \left(\sum_{j=1}^{N_2} g_{G_2}(\Phi_1(\xi), \beta_j) \right) \\ &= \|f_2\|_{\Gamma} \prod_{j=1}^{N_2} |B(\beta'_j, F(\xi))|. \end{aligned} \quad (23)$$

As for q , we use (1) and (11), hence for $\xi \in D_1 \setminus \mathbb{D}$

$$\begin{aligned} \frac{1}{|q(\xi)|} &\leq \frac{1}{|\xi - 1|^2} \exp \left(- \sum_{k=1}^{N_5 - N_2 - 2} \log |B(\gamma_k, \xi)| - \sum_{j=1}^{N_2} \log |B(\beta'_j, \xi)| \right) \\ &= \frac{1}{|\xi - 1|^2} \prod_{k=1}^{N_5 - N_2 - 2} |B(\gamma_k, \xi)|^{-1} \prod_{j=1}^{N_2} |B(\beta'_j, \xi)|^{-1}. \end{aligned} \quad (24)$$

As for Q , we use Bernstein-Walsh estimate for Q on $G_1 \cup \partial G_1$ and that $G_1 \cup \partial G_1 \subset K_u^*$. Therefore, $\|Q\|_{\Gamma} = 1$ and

$$|Q(\Phi_1(\xi))| \leq \|Q\|_{\Gamma} \exp \left(\sum_{j=1}^{N_4} g_{G_2}(\Phi_1(\xi), \zeta_j) \right) \leq \prod_{j=1}^{N_4} |B(\zeta_j, F(\xi))|. \quad (25)$$

Now we are going to multiply together these estimates. Consider the quotients

$$H(\beta', \xi) = H(\xi) := \frac{B(\beta', F(\xi))}{B(\beta', \xi)}$$

for $\beta' \in \mathbb{D}^*$.

First, $\log(H(\xi))'$ is continuous and holomorphic (near 1, using the main branch of the logarithm since $H(1) = 1$), $\log(H)'(1) = 0$ because $F'(1) = 0$. So there exists $C_{14} > 0$ such that $|\log(H(\xi))'| \leq C_{14}|\xi - 1|$ and this is uniformly true for all $\beta' \in \Phi_2^{-1}(Z_2)$ (since $\Phi_2^{-1}(Z_2)$ is compact in \mathbb{D}^*). Now, taking the real part and integrating along radial rays (see also the proof of Lemma 9 in [KN15b]), we get

$$\log |H(\xi)| \leq C_{14} (|\xi| - 1) |\xi - 1|. \quad (26)$$

Applying this estimate for the product of (23), (24), and (25) we can write

$$\begin{aligned}
|f_2(\Phi_1(\xi))| \frac{1}{|q(\xi)|} |Q(\Phi_1(\xi))| &\leq \|f_2\|_\Gamma \prod_{j=1}^{N_2} |B(\beta'_j, F(\xi))| \\
&\cdot \frac{1}{|\xi-1|^2} \prod_{k=1}^{N_5-N_2-2} |B(\gamma_k, \xi)|^{-1} \prod_{j=1}^{N_2} |B(\beta'_j, \xi)|^{-1} \cdot \prod_{j=1}^{N_4} |B(\zeta_j, F(\xi))| \\
&\leq \|f_2\|_\Gamma \frac{1}{\delta_{2,3}^2} \frac{1}{(1+C_8\delta_{2,3})^{N_5-N_2-2-N_4}} \\
&\cdot \exp\left(\sum_{j=1}^{N_2} \log |H(\beta'_j, \xi)| + \sum_{j=1}^{N_4} \log |H(\zeta_j, \xi)|\right) \\
&\leq \|f_2\|_\Gamma n^{4/3} \cdot (1+C_8n^{-2/3})^{-n^{4/5}} \cdot \exp\left(C_{14}(N_2+N_4)\delta_{2,3}2\sqrt{\delta_{2,3}}\right) \\
&\leq \|f_2\|_\Gamma n^{4/3} e^{-C_8/2 n^{2/15}} \exp\left(O\left(n \cdot \frac{1}{n^{2/3}} \frac{1}{n^{1/3}}\right)\right) \\
&= \|f_2\|_\Gamma n^{4/3} e^{-C_8/2 n^{2/15}} O(1) =: \|f_2\|_\Gamma E_{2,3}
\end{aligned}$$

where we used $N_5 - N_2 - 2 = N_4 + [n^{4/5}] - 2$ and (26) at the second step, the definition of $\delta_{2,3}$ and (1) at the third step, again the definition of $\delta_{2,3}$ and that n is large (hence $(1 + \frac{C_8}{n^{2/3}})^{n^{2/3}} \geq e^{C_8/2}$) at the fourth step.

Therefore, the integral over $\Gamma_{2,3}$ can be written as (see (22))

$$\left| \frac{1}{2\pi i} \int_{\Gamma_{2,3}} \frac{(Q \cdot f_2) \circ \Phi_1(\xi)}{\xi - v} \frac{q(v)}{q(\xi)} d\omega \right| \leq \|f_2\|_\Gamma E_{2,3} \frac{8}{\delta_{2,3}} \quad (27)$$

where we used that $|q(v)| \leq 4$ (if $|v| = 1$), and the length of $\Gamma_{2,3}$ is at most 4π .

For $\Gamma_{2,2}$ and $\Gamma_{2,4}$, we apply the same estimate which we detail for $\Gamma_{2,2}$ only. We again start with the integral for $v \in \mathbb{D}$

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{\Gamma_{2,2}} \frac{(Q \cdot f_2) \circ \Phi_1(\xi)}{v - \xi} \frac{q(v)}{q(\xi)} d\xi \right| \\
\leq \frac{1}{2\pi} \int_{\Gamma_{2,2}} 4 \frac{1}{|v - \xi|} |(Q \cdot f_2)(\Phi_1(\xi))| \frac{1}{|q(\xi)|} |d\xi|. \quad (28)
\end{aligned}$$

Since $\xi \in \Gamma_{2,2}$, we can rewrite it in the form $\xi = (1 + \delta) v_1^*$ where $\delta_{2,1} \leq \delta \leq \delta_{2,3}$ (with $v_1^* = v_1^*(\delta_{2,3})$). We use similar steps to estimate f_2 and q and Q .

We use the estimate (23) for $|f_2|$, (25) for $1/|q|$ and (21) for Q . We also use $|\xi| = 1 + \delta$, so

$$\begin{aligned}
|f_2(\Phi_1(\xi))| \frac{1}{|q(\xi)|} |Q(\Phi_1(\xi))| &\leq \|f_2\|_\Gamma \exp\left(\sum_{j=1}^{N_2} \log |H(\beta'_j, \xi)|\right) \\
&\cdot \frac{1}{|\xi-1|^2} \frac{1}{(1+C_8\delta)^{N_5-N_2-2}} \cdot C_4 \exp\left(-C_5(C_{13}/2)^\tau N_3 \delta_{2,3}^{\tau/2}\right) \leq
\end{aligned}$$

which we continue using $1/2\sqrt{\delta} \leq |\xi - 1| \leq 2\sqrt{\delta}$ (see (1)) and $1/(2n) = \delta_{2,1} \leq \delta \leq \delta_{2,3}$ and (26) for the sum. So

$$\leq \|f_2\|_{\Gamma} 8n^2 \exp\left(C_{14}N_2\delta_{2,3}2\sqrt{\delta_{2,3}}\right) C_4 \exp\left(-C_5(C_{13}/2)^{\tau} N_3\delta_{2,3}^{\tau/2}\right) \leq$$

which tends to 0 if $N_3\delta_{2,3}^{\tau/2} \rightarrow \infty$ and we can continue this estimate using $\delta_{2,3} = n^{-2/3}$, so

$$\begin{aligned} &\leq \|f_2\|_{\Gamma} 8C_4 e^{2C_{14}} n^2 \exp\left(-C_5 C_{13}^{\tau} 2^{-3\tau/2} N_3 n^{-\tau/2}\right) \\ &\leq \|f_2\|_{\Gamma} 8C_4 e^{2C_{14}} n^2 \exp\left(-C_5 C_{13}^{\tau} 2^{-3\tau/2} n^{1/12}\right) =: \|f_2\|_{\Gamma} E_{2,2}. \end{aligned}$$

Now continuing (28) we write

$$\leq \frac{1}{2\pi} \int_{\Gamma_{2,2}} 4 \cdot 2n \|f_2\|_{\Gamma} E_{2,2} |d\xi| \leq \frac{2}{\pi} n \delta_{2,3} E_{2,2} \|f_2\|_{\Gamma}$$

where $\delta_{2,3} = n^{-2/3}$ and $nE_{2,2} \rightarrow 0$.

Summarizing these estimates on $\Gamma_{2,1}$, $\Gamma_{2,3}$ and $\Gamma_{2,2}$ (and also on $\Gamma_{2,4}$), and using (3) with the exponential decay of $E_{2,1}, \dots, E_{2,4}$, we have uniformly for $|v| \leq 1$,

$$|r_{2,N_5}(v) - (Q \cdot f_2) \circ \Phi_1(v)| = o(1) \|f\|_{\Gamma}$$

where $o(1)$ tends to 0 as $n \rightarrow \infty$ but it is independent of f and f_2 . Obviously, r_{2,N_5} is a rational function with poles in $\Phi_2^{-1}(Z_2)$ only with $\deg(r_{2,N_5}) = N_5 = (1 + o(1))n$ and therefore we uniformly have for $|v| \leq 1$

$$|r_{2,N_5}(v) - (Q \cdot f_2) \circ \Phi_1(v)| = o(1) \|f\|_{\Gamma},$$

that is,

$$\|r_{2,N_5} - (Q \cdot f_2) \circ \Phi_1\|_{\partial\mathbb{D}} = o(1) \|f\|_{\Gamma} \quad (29)$$

where $o(1)$ tends to 0 as $n \rightarrow \infty$ but it is independent of f . Since 1 is double zero of q , so we have

$$r'_{2,N_5}(1) = ((Q \cdot f_2) \circ \Phi_1)'(1). \quad (30)$$

3.5.4 A similar rational function and final estimates

Consider the “constructed” rational function

$$h(v) := \varphi_{1r}(v) + r_{1,N_5}(v) + r_{2,N_5}(v).$$

Recall that the poles of φ_{1r} are α'_j , $j = 1, \dots, N_1$, the poles of both r_{1,N_5} and r_{2,N_5} are β'_j , $j = 1, \dots, N_2$ and the poles of Q are ζ_j , $j = 1, \dots, N_4$ (counting multiplicities). Hence, this function h has poles at $\alpha'_j = \Phi_1^{-1}(\alpha_j)$ (and with exactly the same multiplicities), and h has poles at β'_j -s and here the multiplicities may change a bit. We use the identity

$$f \circ \Phi_1 = (Q \cdot f + (1 - Q) \cdot f) \circ \Phi_1$$

to calculate the derivatives as follows

$$(((1 - Q) \cdot f) \circ \Phi_1)'(1) = ((1 - Q)' \cdot f)(u_0) \cdot \Phi_1'(1) + ((1 - Q) \cdot f')(u_0) \cdot \Phi_1'(1)$$

where the second term is zero because of the fast decreasing rational function ($Q(u_0) = 1$) and for the first term we can apply the rough Bernstein type inequality (Proposition 4) in the following way ($\|1 - Q\|_\Gamma \leq 2$):

$$|(1 - Q)'(u_0)| \leq \deg(Q) C_1 2 = o(n)$$

where $\deg(Q) = N_4 \leq n^{3/4}$. Here we use that

$$C_{16} := \inf \left\{ \frac{\partial}{\partial n_2(u)} g_{G_2}(u, \beta), \frac{\partial}{\partial n_1(u)} g_{G_1}(u, \alpha) : u \in \Gamma, \alpha \in Z_1, \beta \in Z_2 \right\} > 0 \quad (31)$$

because $g_{G_1}(u, \alpha) = \log |B(\Phi_1^{-1}(u), \Phi_1^{-1}(\alpha))|$, the derivatives of Blaschke factors are bounded away from 0 in modulus ($\Phi_1^{-1}(Z_1)$ is closed and disjoint from the unit circle) and $|\Phi_1'(\cdot)|$ is bounded away from 0. Similarly for $g_{G_2}(\cdot, \cdot)$. There is a uniform upper estimate for the normal derivatives:

$$C_{17} := \sup \left\{ \frac{\partial}{\partial n_2(u)} g_{G_2}(u, \beta), \frac{\partial}{\partial n_1(u)} g_{G_1}(u, \alpha) : u \in \Gamma, \alpha \in Z_1, \beta \in Z_2 \right\} < \infty. \quad (32)$$

Therefore

$$\begin{aligned} & |((1 - Q)' \cdot f)(u_0) \cdot \Phi_1'(1)| \leq \|f\|_\Gamma C_1 n^{3/4} 2 \\ & \leq \|f\|_\Gamma \frac{2n^{3/4} C_1}{\frac{1}{2} n C_{16}} \max \left(\sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j), \sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j) \right) \\ & = o(1) \|f\|_\Gamma \max \left(\sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j), \sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j) \right) \end{aligned} \quad (33)$$

where we used that

$$\frac{1}{2} n C_{16} \leq \max \left(\sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j), \sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j) \right). \quad (34)$$

This way we need to consider $(Q \cdot f) \circ \Phi_1$ only. The derivatives at 1 of the original f and h coincide, because of (6), (18) and (30), so

$$h'(1) = \varphi'_{1r}(1) + r'_{1,N_5}(1) + r'_{2,N_5}(1) = ((Q \cdot f) \circ \Phi_1)'(1). \quad (35)$$

As for the sup norms, we use (6), (17), (29), so we write

$$\|(Q \cdot f) \circ \Phi_1 - h\|_{\partial \mathbb{D}} = o(1) \|f\|_{\partial G_2} \quad (36)$$

where $o(1)$ tends to 0 as $n = \deg(f) \rightarrow \infty$ but it is independent of f , this follows from the considerations after (17) and (29).

Now we apply Borwein-Erdélyi inequality for h as follows:

$$\begin{aligned} |h'(1)| & \leq \|h\|_{\partial \mathbb{D}} \max \left(\sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha'_j), \right. \\ & \quad \left. \sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta'_j) + \sum_{j=1}^{N_5 - N_2 - 2} \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \gamma_j) \right) \end{aligned} \quad (37)$$

where we also used the definition of γ_j 's (see (10)). Now we apply Proposition 6 and (34) with $N_5 - N_2 - 2 = o(n)$ and the uniform upper estimate (32) so we can continue the main estimate (37)

$$\begin{aligned}
&= \|h\|_{\partial\mathbb{D}} \max \left(\sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j), \right. \\
&\quad \left. \sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j) + \sum_{j=1}^{N_5 - N_2 - 2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \Phi_2^{-1}(\gamma_j)) \right) \\
&\leq \|h\|_{\partial\mathbb{D}} (1 + o(1)) \max \left(\sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j), \right. \\
&\quad \left. \sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j) \right) \leq (1 + o(1)) \|f\|_{\Gamma} \max \left(\sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j), \right. \\
&\quad \left. \sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j) \right)
\end{aligned}$$

where in the last step we used (36) and the properties of Q . Here, $o(1)$ tends to 0 as $n = \deg(f) \rightarrow \infty$ but it is independent of f , this follows from the consideration (36) and $N_5 - N_2 - 2 = o(n)$.

Using that $|h'(1)| = |f'(u_0)|$ and summarizing these estimates, we have

$$\begin{aligned}
|f'(1)| &\leq \|f\|_{\Gamma} (1 + o(1)) \\
&\cdot \max \left(\sum_{j=1}^{N_1} \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \alpha_j), \sum_{j=1}^{N_2} \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \beta_j) \right)
\end{aligned}$$

which is the assertion of Theorem 1.

4 Proof of Theorem 2

Here we use an analytic open-up tool to transform the arc setting (z plane) to the curve setting (u plane).

For a Jordan curve Γ , $\text{Int}\Gamma$ denotes the interior of Γ and $\text{Ext}\Gamma$ denotes the exterior of Γ , $\text{Ext}\Gamma := \mathbf{C}_{\infty} \setminus (\Gamma \cup \text{Int}\Gamma)$.

Proposition 9. *Let $\Gamma_0 \subset \mathbf{C}$ be an analytic Jordan arc. Then there exist a rational function F and an analytic Jordan curve Γ such that F is a conformal bijection from $\text{Int}\Gamma$ and from $\text{Ext}\Gamma$ onto $\mathbf{C}_{\infty} \setminus \Gamma_0$.*

This is a special case of [KN15b], Proposition 5 and actually can be established with the Joukowski mapping $z = J(u) = 1/2(u + 1/u)$ and using a suitable linear transformation. This mapping is depicted on figure 3.

Denote the inverse of $z = F(u)$ restricted to $\text{Int}\Gamma$ by $F_1^{-1}(z) = u$ and that restricted to $\text{Ext}\Gamma$ by $F_2^{-1}(z) = u$.

We need the mapping properties of F as regards the normal vectors. For the full details, we refer to [KN15b] p. 879. Briefly, there are exactly two

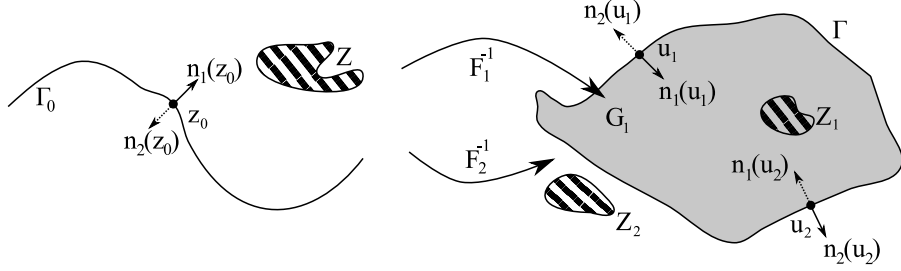


Figure 3: The open-up

$u_1, u_2 \in \Gamma$, $u_1 \neq u_2$ such that $F(u_1) = F(u_2) = z_0$. Denote the normal vectors to Γ pointing outward by $n_2(\cdot)$ and the normal vectors pointing inward by $n_1(\cdot)$. By reindexing u_1 and u_2 and the exchanging $n_1(z_0)$ and $n_2(z_0)$, we may assume that the normal vector $n_2(u_1)$ is mapped by F to the normal vector $n_2(z_0)$. This immediately implies that $n_1(u_1)$, $n_2(u_2)$, $n_1(u_2)$ are mapped by F to $n_1(z_0)$, $n_1(z_0)$, $n_2(z_0)$ respectively.

Moreover, we need to relate the normal derivatives of Green's functions as follows.

Proposition 10. *Using the notations above, for the Green's functions of $G_2 := \text{Ext}\Gamma$ and $G_1 := \text{Int}\Gamma$ and for $b \in \mathbf{C}_\infty \setminus K$ we have*

$$\begin{aligned} \frac{\partial}{\partial n_1(z)} g_{\mathbf{C}_\infty \setminus \Gamma_0}(z, b) &= \frac{\partial}{\partial n_1(u_1)} g_{G_1}(u_1, F_1^{-1}(b)) / |F'(u_1)| \\ &= \frac{\partial}{\partial n_2(u_2)} g_{G_2}(u_2, F_2^{-1}(b)) / |F'(u_2)| \end{aligned}$$

and, similarly for the other side,

$$\begin{aligned} \frac{\partial}{\partial n_2(z)} g_{\mathbf{C}_\infty \setminus \Gamma_0}(z, b) &= \frac{\partial}{\partial n_1(u_2)} g_{G_1}(u_2, F_1^{-1}(b)) / |F'(u_2)| \\ &= \frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, F_2^{-1}(b)) / |F'(u_1)|. \end{aligned}$$

This proposition follows immediately from Proposition 6 from [KN15b] and is a two-to-one mapping analogue of Proposition 6.

Proof of Theorem 2. Use Proposition 9, and consider $f_1(u) := f \circ F(u)$ on the analytic Jordan curve Γ at u_1 (where $F(u_1) = z_0$) and put $G_1 := F_1^{-1}(\mathbf{C}_\infty \setminus \Gamma_0) = \text{Int}\Gamma$ and $G_2 := F_2^{-1}(\mathbf{C}_\infty \setminus \Gamma_0) = \text{Ext}\Gamma$, similarly as above. Obviously, $G = \mathbf{C}_\infty \setminus \Gamma_0$. We have then

$$\begin{aligned} |f'_1(u_1)| &\leq (1 + o(1)) \|f_1\|_\Gamma \\ &\cdot \max \left(\sum_{\beta} \frac{\partial}{\partial n_1(u_1)} g_{G_1}(u_1, F_1^{-1}(\beta)), \sum_{\beta} \frac{\partial}{\partial n_2(u_1)} g_{G_2}(u_1, F_2^{-1}(\beta)) \right) \end{aligned}$$

where β runs through the poles of $f(z)$ (counting multiplicities).

Now use Proposition 10 and $f'_1(u_1) = f'(z_0)F'(u_1)$, hence

$$|f'(z_0)| \leq (1 + o(1)) \|f\|_{\Gamma_0} \cdot \max \left(\sum_{\beta} \frac{\partial}{\partial n_1(z_0)} g_G(z_0, \beta), \sum_{\beta} \frac{\partial}{\partial n_2(z_0)} g_G(z_0, \beta) \right)$$

which we wanted to prove. \square

5 Sharpness - proof of Theorem 3

The idea is as follows. On the unit circle (v plane), we use some special rational functions for which Borwein-Erdélyi inequality is sharp. Then we transfer it back to Γ (u plane) and approximate it with rational functions. In other words, we do the “reconstruction step” in the “opposite direction”.

Recall that $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ and $B(a, v) = \frac{1-\bar{a}v}{v-a}$ is the Blaschke factor with pole at a .

First, we state the cases when we have equality in Borwein-Erdélyi inequality.

Proposition 11. *Suppose h is a Blaschke product with all poles are either inside or outside the unit circle, that is, $h(v) = \prod_{j=1}^n B(\alpha_j, v)$ where all $\alpha_j \in \mathbb{D}$, or $h(v) = \prod_{j=1}^n B(\beta_j, v)$ where all $\beta_j \in \mathbb{D}^*$. Then*

$$|h'(1)| = \|h\|_{\partial\mathbb{D}} \max \left(\sum_{\alpha} \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha), \sum_{\beta} \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta) \right)$$

where the sum with α (or β) is taken over all poles of h in \mathbb{D} (or in \mathbb{D}^* , respectively), counting multiplicities.

This proposition is contained in Borwein-Erdélyi inequality as stated in [BE95] pp. 324-326.

The proof starts as follows. Take n (not necessarily different) points from $\Phi_1^{-1}(Z_1)$, denote them by $\alpha_1, \dots, \alpha_n$ and let

$$h_n(v) := \prod_{j=1}^n B(\alpha_j, v).$$

Obviously $\|h_n\|_{\partial\mathbb{D}} = 1$. Now we “transfer” h_n to u plane.

Let $f_{1,n}(u)$ be the sum of principal parts of $h_n(\Phi_1^{-1}(u))$. And approximate $h_n(\Phi_1^{-1}(u)) - f_{1,n}(u)$ with rational functions with poles outside Γ as follows.

It is immediate that

$$\varphi_e(u) := h_n(\Phi_1^{-1}(u)) - f_{1,n}(u)$$

is holomorphic in $G_1^+ := \{\Phi_1(v) : |v| < 1 + \delta_1\}$.

Fix any $\zeta_0 \in Z_2$ arbitrarily. Take a rational function $w = \psi(u)$ such that $\psi(\zeta_0) = \infty$ and $\deg \psi = 1$. Hence ψ is a conformal mapping from \mathbf{C}_{∞} onto itself.

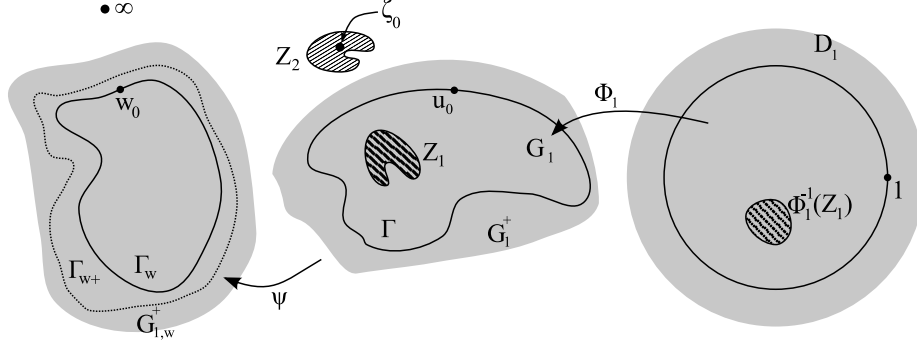


Figure 4: The three planes w , u and v and the mappings

Let $\Gamma_w := \psi(\Gamma)$ and $G_{1,w}^+ := \psi(G_1^+)$ and $w_0 := \psi(u_0)$. Fix a smooth Jordan curve Γ_{w+} such that $\Gamma_w \subset \text{Int}\Gamma_{w+}$, $\Gamma_{w+} \subset G_{1,w}^+$ and $\psi(Z_2) \cap (\Gamma_{w+} \cup \text{Int}\Gamma_{w+}) = \emptyset$. These are depicted on figure 4.

Put

$$N_6 := \lfloor n^{4/5} \rfloor$$

and denote a Fekete polynomial (monic polynomial with zeros at a Fekete point set) of Γ_w with degree N_6 by $P(w)$.

We are going to use the sharpness of Bernstein-Walsh lemma (see, e.g. [Ran95], Theorem 5.5.7 (b), p. 156) therefore we put

$$\begin{aligned} C_{18} &:= \inf \{ g_{\mathbf{C}_\infty \setminus \Gamma_w}(w, \infty) : w \in \Gamma_{w+} \} > 0, \\ C_{19} &:= \sup \{ \tau(w, \infty) : w \in \Gamma_{w+} \} < \infty \end{aligned}$$

where $\tau(.,.)$ denotes the Harnack distance.

It is known that the n -th diameter δ_{N_6} of Γ_w is close to the capacity $\text{cap}(\Gamma_w)$, (see Fekete-Szegő theorem [Ran95], Theorem 5.5.2 on page 153), since $\delta_{N_6} \rightarrow \text{cap}(\Gamma_w)$. Hence N_6 is large, then $C_{19} \log \frac{\delta_{N_6}}{\text{cap}(\Gamma_w)} < C_{18}/2$ and $g(w, \infty) - \tau(w, \infty) \log \frac{\delta_{N_6}}{\text{cap}(\Gamma_w)} > C_{18}/2$. This way we have for large N_6 that

$$|P(w)| \geq \|P\|_{\Gamma_w} \exp(N_6 C_{18}/2), \quad (w \in \Gamma_{w+}). \quad (38)$$

Let

$$q(w) := P(w) (w - w_0)^2,$$

this q is a polynomial with degree $N_6 + 2$ and w_0 is (at least) a double zero. Introduce

$$f_{2,w}(w) := \frac{1}{2\pi i} \int_{\Gamma_{w+}} \frac{\varphi_e(\psi^{-1}(w))}{q(\xi)} \frac{q(w) - q(\xi)}{w - \xi} d\xi$$

where $w \in \text{Int}\Gamma_{w+}$ and this $f_{2,w}(.)$ is a polynomial with degree $N_6 + 2$. Here, $f'_{2,w}(w_0) = \varphi'_e(u_0)/\psi'(u_0)$.

The error of interpolation on the w plane is

$$\varphi_e(\psi^{-1}(w)) - f_{2,w}(w) = \frac{1}{2\pi i} \int_{\Gamma_{w+}} \frac{\varphi_e(\psi^{-1}(\xi))}{\xi - w} \frac{q(w)}{q(\xi)} d\xi,$$

where $w \in \text{Int}\Gamma_{w+}$. It can be estimated from above as follows if $w \in \Gamma_w$

$$\begin{aligned} |\varphi_e(\psi^{-1}(w)) - f_{2,w}(w)| &\leq \frac{1}{2\pi} \int_{\Gamma_{w+}} \left| \frac{\varphi_e(\psi^{-1}(\xi))}{\xi - w} \right| \left| \frac{q(w)}{q(\xi)} \right| |d\xi| \\ &\leq \frac{1}{2\pi} \int_{\Gamma_{w+}} |\varphi_e(\psi^{-1}(\xi))| \left| \frac{P(w)}{P(\xi)} \right| C_{20} \frac{1}{|\xi - w| |\xi - w_0|^2} |d\xi| \quad (39) \end{aligned}$$

where $C_{20} > 0$ (actually $C_{20} = \text{diam}(\Gamma_{w+})^2$) and we continue this estimate later. Here $|\varphi_e(\psi^{-1}(\xi))|$, $\xi \in \Gamma_{w+}$ can be estimated using that $h_n(v)$ is a Blaschke product with all poles in the unit disk, therefore $\|h_n \circ \Psi_1^{-1} \circ \psi^{-1}\|_{\Gamma_{w+}} \leq 1$ and using the Gonchar-Grigorjan type estimate (see Theorem 1 in [KN15a]) on $\text{Int}\Gamma$ for $h_n \circ \Phi_1^{-1}$ which implies that $\|f_{1,n}\|_{\Gamma} \leq C_6(\Gamma) \log(n) \|h_n \circ \Phi_1^{-1}\|_{\Gamma} = C_6(\Gamma) \log(n)$ and, since $f_{1,n}$ has poles in $\text{Int}\Gamma$, the maximum principle yields $\|f_{1,n}\|_{\Gamma_w} \geq \|f_{1,n}\|_{\Gamma_{w+}}$. Hence

$$\begin{aligned} |\varphi_e(\psi^{-1}(\xi))| &\leq \|h_n \circ \Psi_1^{-1} \circ \psi^{-1}\|_{\Gamma_{w+}} + \|f_{1,n}\|_{\Gamma_{w+}} \\ &\leq 1 + C_6(\Gamma) \log(n) = O(\log(n)) \quad (40) \end{aligned}$$

where n is large enough (depending on Γ only). We also use

$$\frac{C_{20}}{2\pi} \int_{\Gamma_{w+}} \frac{1}{|\xi - w| |\xi - w_0|^2} |d\xi| \leq C_{21}$$

for some constant $C_{21} > 0$ depending on Γ and Γ_{w+} , since $w \in \Gamma_w$, and Γ_{w+} is fixed and is from positive distance from Γ_w . Finally, we estimate $|q(w)/q(\xi)|$ using (38) (when $w \in \Gamma_w$, $\xi \in \Gamma_{w+}$), whence

$$\left| \frac{q(w)}{q(\xi)} \right| \leq \frac{\|P\|_{\Gamma_w}}{\|P\|_{\Gamma_w} e^{N_6 C_{18}/2}} = e^{-N_6 C_{18}/2}.$$

Putting these estimates together, we can finish (39)

$$\leq O\left(\log(n) e^{-N_6 C_{18}/2}\right).$$

Now substituting $w = \psi(u)$, we can write

$$\|\varphi_e - f_{2,w} \circ \psi\|_{\Gamma} \leq O\left(\log(n) e^{-N_6 C_{18}/2}\right) \quad (41)$$

where δ_3 and the constant in $O(\cdot)$ is independent of h_n and n and depends only on Γ and Γ_{w+} only.

Finally, let

$$f_n(u) := f_{1,n}(u) + f_{2,w}(\psi(u))$$

this is a rational function with poles in $\text{Int}\Gamma$ with total order n and one pole in the exterior of Γ (at ζ_0) with order $N_6 + 2 = O(n^{4/5}) = o(n)$. Moreover

$$|f'_n(u_0)| = |f'_{1,n}(u_0) + (\varphi'_e(u_0)/\psi'(u_0))\psi'(u_0)| = |h'_n(1)|$$

since $|\Phi'_1(1)| = 1$. We know that $\|h_n\|_{\partial\mathbb{D}} = 1$ and using (41),

$$\begin{aligned}\|f_n\|_\Gamma &= \|f_{1,n} + f_{2,n} \circ \psi\|_\Gamma = \|f_{1,n} + \varphi_e + f_{2,n} \circ \psi - \varphi_e\|_\Gamma \\ &= \|h_n \circ \Phi_1^{-1} + f_{2,n} \circ \psi - \varphi_e\|_\Gamma = 1 \pm O\left(\log(n)e^{-N_6 C_{18}/2}\right)\end{aligned}$$

therefore $\|f_n\|_\Gamma \leq (1 + o(1))\|h_n\|_{\partial\mathbb{D}}$.

We use Proposition 11 for h_n , hence

$$\begin{aligned}|f'_n(u_0)| &= |h'_n(1)| \\ &= \|h_n\|_{\partial\mathbb{D}} \max\left(\sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha), \sum_\beta \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta)\right) \\ &\geq (1 - o(1))\|f_n\|_\Gamma \max\left(\sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha), \sum_\beta \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta)\right)\end{aligned}$$

where actually the second term in the max is void (h_n has poles only inside the unit disk) and by Proposition 6, so

$$\begin{aligned}\max\left(\sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha), \sum_\beta \frac{\partial}{\partial n_2(1)} g_{\mathbb{D}^*}(1, \beta)\right) &= \sum_\alpha \frac{\partial}{\partial n_1(1)} g_{\mathbb{D}}(1, \alpha) \\ &= \sum_\alpha \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \Phi_1(\alpha)) \\ &= \max\left(\sum_\alpha \frac{\partial}{\partial n_1(u_0)} g_{G_1}(u_0, \Phi_1(\alpha)), (N_6 + 2) \frac{\partial}{\partial n_2(u_0)} g_{G_2}(u_0, \zeta_0)\right)\end{aligned}$$

in the last step we used that the first term in the max is $O(n)$ and the second one is $o(n)$, hence if n is large depending on Γ , then the last equality holds.

This finishes the proof of Theorem 3.

6 Acknowledgement

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References

- [Bar92] Mirosław Baran, *Bernstein type theorems for compact sets in \mathbf{R}^n* , J. Approx. Theory **69** (1992), no. 2, 156–166. MR 1160251 (93e:41021)

- [BE95] Peter Borwein and Tamás Erdélyi, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, vol. 161, Springer-Verlag, New York, 1995. MR 1367960 (97e:41001)
- [BE96] ———, *Sharp extensions of Bernstein's inequality to rational spaces*, *Mathematika* **43** (1996), no. 2, 413–423 (1997). MR 1433285 (97k:26014)
- [DK07] V. N. Dubinin and S. I. Kalmykov, *A majorization principle for meromorphic functions*, *Mat. Sb.* **198** (2007), no. 12, 37–46, English translation in *Sb. Math.* 198 (2007), no. 11-12, 1737–1745. MR 2380804 (2009c:30074)
- [Dub12] V. N. Dubinin, *Methods of geometric function theory in classical and modern problems for polynomials*, *Uspekhi Mat. Nauk* **67** (2012), no. 4(406), 3–88, English translation in *Russian Math. Surveys* 67 (2012), no. 4, 599–684. MR 3013845
- [GG76] A. A. Gončar and L. D. Grigorjan, *Estimations of the norm of the holomorphic component of a meromorphic function*, *Math. USSR-Sb.* **28** (1976), no. 4, 571–575, Russian original in *Mat. Sb. (N.S.)* 99(141) (4) 634–638. MR 0417417
- [KN15a] S. I. Kalmykov and B. Nagy, *On estimate of the norm of the holomorphic component of a meromorphic function in finitely connected domain*, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **440** (2015), no. Analiticheskaya Teoriya Chisel i Teoriya Funktsii. 30, 123–137.
- [KN15b] ———, *Polynomial and rational inequalities on analytic Jordan arcs and domains*, *J. Math. Anal. Appl.* **430** (2015), no. 2, 874–894. MR 3351986
- [Luk04] A. L. Lukashov, *Inequalities for the derivatives of rational functions on several intervals*, *Izv. Ross. Akad. Nauk Ser. Mat.* **68** (2004), no. 3, 115–138, English translation in *Izv. Math.* 68 (2004), no. 3, 543–565. MR 2069196 (2006j:26010)
- [Mar77] A. I. Markushevich, *Theory of functions of a complex variable. Vol. I, II, III*, english ed., Chelsea Publishing Co., New York, 1977, Translated and edited by Richard A. Silverman. MR 0444912 (56 #3258)
- [MMR94] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, World Scientific Publishing Co., Inc., River Edge, NJ, 1994. MR 1298187 (95m:30009)
- [Nag05] Béla Nagy, *Asymptotic Bernstein inequality on lemniscates*, *J. Math. Anal. Appl.* **301** (2005), no. 2, 449–456. MR 2105685 (2006c:41019)
- [NT05] Béla Nagy and Vilmos Totik, *Sharpening of Hilbert's lemniscate theorem*, *J. Anal. Math.* **96** (2005), 191–223. MR 2177185 (2006g:30008)
- [NT13] ———, *Bernstein's inequality for algebraic polynomials on circular arcs*, *Constr. Approx.* **37** (2013), no. 2, 223–232. MR 3019778

- [NT14] ———, *Riesz-type inequalities on general sets*, J. Math. Anal. Appl. **416** (2014), no. 1, 344–351. MR 3182764
- [Pom92] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992. MR 1217706 (95b:30008)
- [Ran95] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR 1334766 (96e:31001)
- [SL68] V. I. Smirnov and N. A. Lebedev, *Functions of a complex variable: Constructive theory*, Translated from the Russian by Scripta Technica Ltd, The M.I.T. Press, Cambridge, Mass., 1968. MR 0229803 (37 #5369)
- [ST97] Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997, Appendix B by Thomas Bloom. MR 1485778 (99h:31001)
- [Tot01] Vilmos Totik, *Polynomial inverse images and polynomial inequalities*, Acta Math. **187** (2001), no. 1, 139–160. MR 1864632 (2002h:41017)
- [Tot10] ———, *Christoffel functions on curves and domains*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 2053–2087. MR 2574887 (2011b:30006)
- [Wal65] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX, American Mathematical Society, Providence, R.I., 1965. MR 0218588 (36 #1672b)
- [Wid69] Harold Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Advances in Math. **3** (1969), 127–232 (1969). MR 0239059 (39 #418)

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